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Non-linear wave propagation in a relaxing gas

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We consider the propagation of waves of small finite amplitude ϵ in a gas whose internal energy is characterized by two temperatures T (translational) and T_i (internal) in the form $e = C_{v_f}T + C_{v_i}T_i$, and T_i is governed by a rate equation $dT_i/dt = (T - T_i)/\tau$. By means of approximations appropriate for a wave advancing into an undisturbed region $x > 0$, we show that to order $\epsilon\delta$, the equation satisfied by velocity takes the non-linear form

$$\left(\tau\frac{\partial}{\partial t}+1\right)\left\{\frac{\partial u}{\partial t}+\left(a_1+\frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x}-\tfrac{1}{2}\lambda\frac{\partial^2 u}{\partial x^2}\right\} = (a_1 - a_0)\frac{\partial u}{\partial x},$$

where a_1, a_0 are the frozen and equilibrium speeds of sound in the undisturbed region, $\delta = \frac{1}{2}(1-(a_0^2/a_1^2))$, and λ is the diffusivity of sound due to viscosity and heat conduction (λ may be neglected except when discussing the fine structure of a discontinuity). Some numerical solutions of this model equation are given.

When ϵ is small compared with δ, it is also possible to construct a solution for the flow produced by a piston moving with a constant velocity by means of a sequence of matched asymptotic expansions. The limit reached for large times for either compressive or expansive pistons is the expected non-linear solution of the exact equations. For a certain range of advancing piston speeds, this is a fully dispersed wave with velocity U in the range $a_0 < U < a_1$. If $U > a_1$ the solution is discontinuous, and indeterminate in the absence of viscosity; a singular perturbation technique based on λ is then used to determine the structure of the wave head.

1. Introduction

The linearized theory of wave propagation in a gas with a single relaxing internal degree of freedom has been discussed by Chu (1958), Vincenti (1959), Clarke (1960, 1961), Moore & Gibson (1960) and Lick (1967). The thermodynamic model used by all these authors is one in which the internal energy per unit mass is characterized by the translational temperature $T = p/\rho R$ and an internal temperature T_i (which might, for instance, be a measure of the energy in the molecular vibrational states if their excitation were the physical process under discussion), by means of the equation

$$e = C_{v_f}T + C_{v_i}T_i, \tag{1.1}$$

where the specific heats C_{v_f}, C_{v_i} of frozen and relaxing modes are supposed constant in the temperature range under consideration, and the approach of T_i to its equilibrium value T is described by a *rate equation*

$$\frac{dT_i}{dt} = \frac{T - T_i}{\tau}. \tag{1.2}$$

τ here is the relaxation time, treated as a constant by the above authors, although, as noted in §3, their (and our) analysis remains valid if τ is, more realistically, regarded as a function of p and ρ, provided the derivatives $\partial\tau/\partial p$, $\partial\tau/\partial\rho$ are not too large. For adiabatic flow, for which the energy equation is

$$\frac{de}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right) = 0, \tag{1.3}$$

elimination of e, T, T_i leads to the equation

$$\rho\tau^* \frac{d}{dt}\left[\frac{1}{\rho}\left(\frac{dp}{dt} - a_f^2\frac{d\rho}{dt}\right)\right] + \frac{dp}{dt} - a_e^2\frac{d\rho}{dt} = 0, \tag{1.4}$$

in which p, ρ are pressure and density,

$$\tau^* = \left(\frac{\gamma_e - 1}{\gamma_f - 1}\right)\tau,$$

and

$$a_f = (\gamma_f p/\rho)^{\frac{1}{2}}, \quad a_e = (\gamma_e p/\rho)^{\frac{1}{2}} \tag{1.5}$$

are respectively the frozen and equilibrium sound speeds, γ_f and γ_e being $1 + RC_{v_f}^{-1}$, $1 + RC_{v_e}^{-1}$ respectively, where R is the universal gas constant and $C_{v_e} = C_{v_f} + C_{v_i}$ is the specific heat for a process for which thermodynamic equilibrium prevails throughout. It will be noted that a_f is always greater than a_e.

When the acoustic approximations appropriate to one-dimensional flow, namely

$$\partial p/\partial x = -\rho\,\partial u/\partial t, \quad \partial\rho/\partial t = -\rho\,\partial u/\partial x,$$

$$a_f = \text{constant} = a_1, \quad a_e = \text{constant} = a_0,$$

are introduced in (1.4), and second-order terms excluded, the equation takes the linear form

$$\tau^*\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - a_1^2\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2 u}{\partial t^2} - a_0^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.6}$$

studied by the above-mentioned authors. This belongs to a general class of equations studied by Whitham (1959), and the character of the solution is well understood. At times small compared with τ^*, the highest order terms govern the propagation of disturbances, which therefore travel at the frozen speed a_1, but at times large compared with τ^* it is the equilibrium speed a_0 which governs the motion. For the case of a piston set impulsively in motion at $t = 0$ in a semi-infinite column of gas $x > 0$, the two limiting solutions are

$$\left.\begin{aligned} t/\tau^* < 1:&\quad u = u_0\, e^{-\delta x/a_1\tau^*} \quad \text{for} \quad x \leqslant a_1 t;\\ t/\tau^* > 1:&\quad u = u_0\, \text{erfc}\left[\left(t - \frac{x}{a_0}\right)\Big/ 2\sqrt{\left(\frac{\delta\tau^* x}{a_0}\right)}\right],\end{aligned}\right\} \tag{1.7}$$

where u_0 is the piston speed, and

$$\delta = \frac{1}{2}\left(\frac{\gamma_f-\gamma_e}{\gamma_f}\right) = \frac{1}{2}\left\{1-\left(\frac{a_0}{a_1}\right)^2\right\}, \tag{1.8}$$

which is assumed to be a small quantity. (For vibrational relaxation of a diatomic gas $\delta = \frac{2}{49}$.)

Thus linearized theory predicts that the head of the disturbance at large times will be diffusive in character, centred on the equilibrium characteristic $x = a_0 t$, and occupying a region whose width grows parabolically with time or with distance from the origin. This is not, however, the asymptotic behaviour that would be expected physically for either a compressive or an expansive wave. For the former, Lighthill (1956) has pointed out that the full non-linear equations possess a solution in which the velocity rises from zero to a small finite value (that of a compressive piston) through a wave form that can propagate without change of shape, and is continuous provided its velocity (which depends on that of the piston) lies between a_0 and a_1. And for an expansive piston, an asymptotic examination of the full equations of motion leads one to the centred expansion fan appropriate to the equilibrium sound speed a_0 as $t \to \infty$. Both these asymptotic results require the retention of convective terms in the equation of motion, and the present work is aimed at finding an improved approximation to the equations which will retain the essential non-linearity of the problem and still permit analytic treatment, in order to study the evolution of the wave form from that of linear theory to its asymptotic state.

The method we adopt is analogous to that used by Lighthill (1956, p. 250) to treat the influence of viscosity and heat conduction in finite disturbances propagating into undisturbed gas. Essentially, we exploit the fact that for such a unidirectional disturbance the operator $\partial/\partial t + a\,\partial/\partial x$, where a is *either* of the sound speeds a_1, a_0, is small compared with the separate operators $\partial/\partial t$, $\partial/\partial x$. This is true provided both δ and the amplitude of the velocity disturbance, denoted non-dimensionally by $\epsilon = u_0/a_1$, are small compared with unity. The resulting equation can be written as

$$\tau^* \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t} + \left(a_1 + \frac{\gamma_f+1}{2}u\right)\frac{\partial u}{\partial x}\right] + \left\{\frac{\partial u}{\partial t} + \left(a_0 + \frac{\gamma_e+1}{2}u\right)\frac{\partial u}{\partial x}\right\} = 0. \tag{1.9}$$

An equation of this form might well have provided a plausible physical model even if it had not been capable of analytical derivation. It is exact in the two limiting cases $\tau = 0$, $\tau = \infty$, corresponding to non-relaxing flow in frozen and equilibrium conditions respectively (because for flow into an undisturbed region $u_t + (u+a)\,u_x = 0$, and $a = a_0 + \frac{1}{2}(\gamma-1)\,u$); and the form

$$\tau^* \frac{\partial}{\partial t} L_f u + L_e u = 0, \tag{1.10}$$

where L_f, L_e denote frozen and equilibrium operators, is that which might have been expected by analogy with the linear result (1.6).

Two limiting forms of (1.9) may be noted here:

(i) On omission of the non-linear terms it reduces to the telegraph equation

$$\tau^* \frac{\partial}{\partial t}(u_t + a_1 u_x) + u_t + a_0 u_x = 0, \tag{1.11}$$

the connexion of which with relaxing gas motion was first noted by Moore & Gibson (1960). The asymptotic forms of the solution for $t/\tau^* \gg 1$ and $\ll 1$ are identical with those of the full linear equation already quoted (1.7), so (1.11) is not adequate to discuss the development of the flow at very large times even when the velocity is extremely small.

(ii) Very close to equilibrium, i.e. for $t/\tau^* \gg 1$, we expect the flow to be given to a first approximation by equating the curly bracket on the left of (1.9) to zero.

To improve this approximation, we use it to evaluate the square bracket, which then becomes simply $\tau^*[\partial/\partial t(a_1 - a_0)\, \partial u/\partial x]$, and since $\partial/\partial t \doteqdot -a_1 \partial/\partial x$, this is $-a_1^2 \delta\tau^* \partial^2 u/\partial x^2$, so the equation becomes

$$\frac{\partial u}{\partial t} + \left(a_0 + \frac{\gamma_f + 1}{2} u\right) \frac{\partial u}{\partial x} = a_1^2 \delta\tau^* \frac{\partial^2 u}{\partial x^2} \tag{1.12}$$

in this region. This is Burgers' equation, which is satisfied, as Lighthill (1956) and Lagerstrom, Cole & Trilling (1949) have shown, by small motions of a viscous heat-conducting gas, in which case $a_1^2\delta\tau^*$ is replaced by the 'diffusivity of sound', λ say. This confirms the fact that sufficiently close to equilibrium the effect of relaxation can be represented by a bulk viscosity, as had been assumed by Jones (1964) in discussing the structure of a rarefaction wave at large times. [It is of course well known for sound waves (cf. Landau & Lifschitz 1959, pp. 304–9) and the expression (1.7) is just that obtainable from viscous acoustic theory.]

In § 2, we outline the limiting solutions for large time of the exact equations, which for this purpose are (1.4) together with the equations of momentum and continuity. Then, in § 3, we derive the model equation (1.9) as an approximation to the exact equations, and note that the limiting forms of its solution correspond to the exact limits of § 2.

In § 4, we use the model equation to treat the motion produced in a semi-infinite column of gas initially at rest when a piston is set in motion at one end. The complete solution is found by matched asymptotic expansion techniques and it is shown that the expected limiting forms are indeed attained at large times.

Section 5 contains a discussion of weak discontinuities in a relaxing gas. The shock relations are derived by finding a continuous solution of a model equation similar to (1.9) but including viscous terms and then letting the viscosity tend to zero. Using these shock relations, the solution to the impulsive piston problem can be found.

2. Equations of motion of one-dimensional relaxing gas flow

To discuss the flow we must add to (1.4) the continuity and momentum equations

$$\frac{d\rho}{dt} + \rho u_x = 0, \tag{2.1}$$

$$\rho \frac{du}{dt} + p_x = 0. \tag{2.2}$$

Substitution of (2.1) puts (1.4) into the form

$$\left(\tau^* \frac{d}{dt} + 1\right)\left[\frac{1}{\rho}\left(\frac{dp}{dt} + \gamma_f p u_x\right)\right] = (\gamma_f - \gamma_e)\frac{p u_x}{\rho} \tag{2.3}$$

and these three provide a convenient form for the governing equations. We wish to discuss wave propagation into a semi-infinite region $x > 0$ of gas initially at rest at pressure p_0 and density ρ_0, and shall denote the undisturbed speeds of sound by a_1, a_0 where

$$a_1^2 = \frac{\gamma_f p_0}{\rho_0}, \quad a_0^2 = \frac{\gamma_e p_0}{\rho_0}. \tag{2.4}$$

We note first the asymptotic solution as $t \to \infty$ for the motion produced by a piston moving at constant velocity c.

(i) *Compressive piston*, $c > 0$

In this case, provided the piston speed is not greater than

$$\frac{2}{\gamma_e+1}\left(\frac{a_1^2-a_0^2}{a_1}\right),$$

the asymptotic wave form is the continuous 'fully dispersed shock' described by Broer (1951) and Lighthill (1956), propagating at a speed U intermediate between a_1 and a_0, the speed U being related to a_0 and c by the usual Rankine–Hugoniot relationship appropriate to thermodynamic equilibrium. If, however, c exceeds the stated value, the velocity jumps discontinuously at the wave head, which is followed by a region of continuous velocity variation produced by relaxation. The form of the wave is found by writing p, ρ and u as functions of the single independent variable

$$\xi = x - Ut, \tag{2.5}$$

the boundary conditions being $u = 0$, c at $\xi = \pm\infty$, with $du/d\xi = 0$ at both limits, and $p = p_0$, $\rho = \rho_0$ at $\xi = \infty$.

Equations (2.1) and (2.2) can then be integrated to the usual conservation forms

$$\rho(U-u) = \rho_0 U, \tag{2.6}$$

$$p = p_0 + \rho_0 uU. \tag{2.7}$$

If we make the further change of variable

$$d\xi = \left(\frac{\rho_0}{\rho}\right) dz, \tag{2.8}$$

(2.3) becomes, on substitution for p, ρ from (2.6), (2.7) and integration

$$\tau^*[a_1^2 - U^2 + (\gamma_f+1)uU]\frac{du}{dz} = \tfrac{1}{2}(\gamma_e+1)u\left[u - \frac{2}{\gamma_e+1}\left(U - \frac{a_0^2}{U}\right)\right], \tag{2.9}$$

where the condition $du/dz = 0$ at $u = 0$ has been applied. Since du/dz also vanishes at $u = c$, we must have

$$\frac{2}{\gamma_e+1}\left(U - \frac{a_0^2}{U}\right) = c, \quad \text{i.e. } U = \left(\frac{\gamma_e+1}{4}\right)c + a_0\left[1 + \left(\frac{\gamma_e+1}{4}\frac{c}{a_0}\right)^2\right]^{\frac{1}{2}}, \tag{2.10}$$

which, as stated, is just the *equilibrium* Rankine–Hugoniot relation between U and c. Provided $U < a_1$, a second integration is possible and gives the shock profile

$$z = \left(\frac{2}{\gamma_e+1}\right)\left(\frac{\tau^*}{c}\right)[A\ln(c-u) - B\ln u], \tag{2.11}$$

where
$$A = a_1^2 - U^2 + (\gamma_f+1)cU, \quad B = a_1^2 - U^2.$$

If $U > a_1$, the profile given by this expression turns back on itself as shown in figure 1, and there is a discontinuous jump in velocity from the value $u = 0$ to a point on the upper half of the profile. This takes place at a constant value of T_i, namely the upstream value T_0. (If viscosity and heat conduction are taken into account, the jump is replaced by a continuous but rapid change in velocity on a length scale of order λ/a, where λ is the diffusivity of sound.) In terms of ξ, the profile shape is

$$\xi = \frac{2\tau^*}{(\gamma_e+1)c}\left[\left(1-\frac{c}{U}\right)A\ln(c-u)-B\ln u\right]-2\tau^*\left(\frac{\gamma_f+1}{\gamma_e+1}\right)u. \tag{2.12}$$

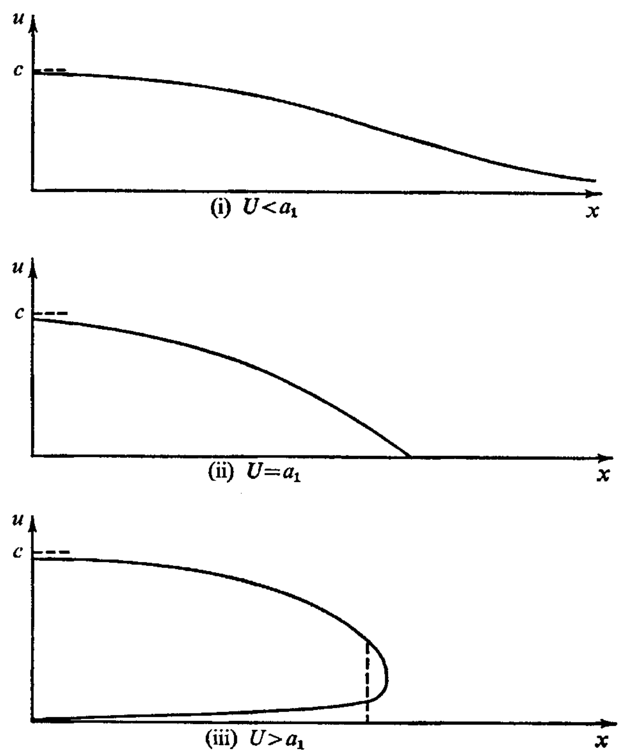

FIGURE 1. Steady state solutions of equation (2.11).

It will be noted that the limiting form of either expression, as $c/a_1 \to 0$, is

$$\frac{4}{\gamma_e+1}\left(\frac{a_1^2\tau^*\delta}{c}\right)\ln\left(\frac{c-u}{u}\right). \tag{2.13}$$

This is identical with the profile for a Taylor shock in a fluid of diffusivity $a_1^2\tau^*\delta$.

(ii) *Expansive piston,* $c < 0$

To discuss this motion, which has been treated numerically by Wood & Parker (1958) and Appleton (1960), we change to independent variables

$$\eta = x/t, \quad \zeta = t \tag{2.14}$$

and look for a solution in which the pressure, density and velocity p, ρ, u in (2.1)–(2.3) are written as expansions

$$p(\eta)+\frac{1}{\zeta}p_1(\eta)+\ldots, \quad \rho(\eta)+\frac{1}{\zeta}\rho_1(\eta)+\ldots, \quad u(\eta)+\frac{1}{\zeta}u_1(\eta)+\ldots. \tag{2.15}$$

(A similar expansion was proposed by Napolitano (1962) for the corresponding two-dimensional steady problem.) The lowest order terms (those in $1/\zeta$) give the equations

$$(u-\eta)\,d\rho+\rho\,du = 0, \quad \rho(u-\eta)\,du+dp = 0, \quad (u-\eta)\,dp+\gamma_e p\,du = 0, \tag{2.16}$$

which are readily integrated to show $p/p_0 = (\rho/\rho_0)^{\gamma_e}$, whence the structure of the expansion fan is found as

$$u = \begin{Bmatrix} 0, & \dfrac{x}{t} > a_0, \\ \dfrac{2}{\gamma_e+1}\left(\dfrac{x}{t}-a_0\right), & a_0+\left(\dfrac{\gamma_e+1}{2}\right)c < \dfrac{x}{t} < a_0, \\ c, & \dfrac{x}{t} < a_0+\left(\dfrac{\gamma_e+1}{2}\right)c, \end{Bmatrix} \tag{2.17}$$

which is just that of a centred expansion fan in a gas with the equilibrium value of γ. The equations (2.16) are in fact those that would be found exactly if (2.3) had been taken in the equilibrium form $dp/dt+\gamma_e p u_x = 0$ at the outset.

3. Derivation of an approximate equation for weak waves travelling in the positive direction

We wish to discuss motions in which the wave produced by a piston is advancing into gas at rest at pressure p_0, for the case in which the scale ϵ say of the non-dimensional piston velocity u_0/a_1 is at most of the same order as the lagging energy parameter δ already defined by (1.8). The dimensionless perturbations u/a_1, $(p-p_0)/p_0$ and $(\rho-\rho_0)/\rho_0$ in flow variables will then be of this order, and the results of the linearized theory summarized in § 1 lead us to expect that these perturbations will be slowly varying functions of $x-at$, where a is a (varying) sound speed intermediate between the frozen and equilibrium values a_1 and a_0. If the major dependence on x, t of the stated flow perturbations is of this form, it follows that the operator $\partial/\partial t+a_1\partial/\partial x$, when applied to them, is to a first approximation the same as $(a_1-a)\,\partial/\partial x$. This is of order δ/τ provided the relaxation time τ is the appropriate time-scale for the motion: a piston with finite acceleration might introduce a further time scale as well, but for the present we shall mainly concern ourselves with pistons impulsively set into motion with constant velocity.

Using the above approximation, we now proceed to reduce (2.1), (2.2) and (2.3) to a single equation for u. The problem contains two independent small parameters ϵ and δ but since the steady fully dispersed shock is possible only if $\epsilon < 4\delta/(\gamma_e+1)$ we shall only consider the case $\epsilon \lesssim \delta \ll 1$. We introduce non-dimensional scaled variables and derive the approximate equation on the assumption that the operator $(\partial/\partial t + a_1\partial/\partial x)$ is $O(\delta/\tau)$ and then neglect terms in ϵ^3, $\epsilon^2\delta$ (but retain all powers of δ). We first define non-dimensional variables as follows:

$$\begin{aligned} u &= a_1\epsilon u', \\ p &= p_0(1+\epsilon\gamma_f p'), \\ \rho &= \rho_0(1+\epsilon\rho'), \\ x &= a_1\tau^* x', \\ t &= \tau^* t'. \end{aligned}$$

Equations (2.1), (2.2) and (2.3) are then:

$$\frac{\partial\rho'}{\partial t'} + \frac{\partial u'}{\partial x'} + \epsilon\left(\frac{\partial}{\partial x'}(\rho' u')\right) = 0, \tag{3.1}$$

$$\frac{\partial u'}{\partial t'} + \frac{\partial p'}{\partial x'} + \epsilon\left[\rho'\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'}\right] + \epsilon^2\rho' u'\frac{\partial u'}{\partial x'} = 0, \tag{3.2}$$

$$\left(\frac{\partial}{\partial t'} + 1 + \epsilon u'\frac{\partial}{\partial x'}\right)\left(\frac{1}{1+\epsilon\rho'}\left(\frac{\partial p'}{\partial t'} + \epsilon u'\frac{\partial p'}{\partial x'} + \frac{\partial u'}{\partial x'} + \gamma_f\epsilon p'\frac{\partial u'}{\partial x'}\right)\right) = \frac{2\delta(1+\epsilon\gamma_f p')}{1+\epsilon\rho'}\frac{\partial u'}{\partial x'}. \tag{3.3}$$

If we write the equations in terms of the independent variables $Y = x' - t'$ and $T' = t'$ we see that

$$\begin{aligned} \frac{\partial}{\partial t} + a_1\frac{\partial}{\partial x} &= \frac{1}{\tau^*}\left(\frac{\partial}{\partial t'} + \frac{\partial}{\partial x'}\right) \\ &= \frac{1}{\tau^*}\left(\frac{\partial}{\partial T'}\right)_Y. \end{aligned}$$

Assuming that this operator is of $O(\delta/\tau)$ is equivalent to stretching T' by putting $T' = T/\delta$ and then taking the operator $\partial/\partial T$ to be of $O(1)$. In the following work the variables used will be non-dimensional unless stated otherwise and so, dropping dashes and writing equations (3.1), (3.2) and (3.3) in terms of Y and T, we obtain

$$-\frac{\partial\rho}{\partial Y} + \frac{\partial u}{\partial Y} + \delta\frac{\partial\rho}{\partial T} + \epsilon\frac{\partial}{\partial Y}(\rho u) = 0, \tag{3.4}$$

$$-\frac{\partial u}{\partial Y} + \frac{\partial p}{\partial Y} + \delta\frac{\partial u}{\partial T} + \epsilon\left[-\rho\frac{\partial u}{\partial Y} + u\frac{\partial u}{\partial Y}\right] + \epsilon\delta\rho\frac{\partial u}{\partial T} + \epsilon^2\rho u\frac{\partial u}{\partial Y} = 0, \tag{3.5}$$

$$\left[1 - \frac{\partial}{\partial Y} + \delta\frac{\partial}{\partial T} + \epsilon u\frac{\partial u}{\partial Y}\right]\left[\left(\frac{1}{1+\epsilon\rho}\right)\left(-\frac{\partial p}{\partial Y} + \frac{\partial u}{\partial Y} + \delta\frac{\partial p}{\partial T} + \epsilon u\frac{\partial p}{\partial Y} + \epsilon\gamma_f p\frac{\partial u}{\partial Y}\right)\right] = 2\delta\left(\frac{1+\epsilon\gamma_f p}{1+\epsilon\rho}\right)\frac{\partial u}{\partial Y}. \tag{3.6}$$

The first-order approximation to these equations leads to

$$\left.\begin{aligned} \rho &= u + O(\epsilon, \delta), \\ p &= u + O(\epsilon, \delta), \end{aligned}\right\} \tag{3.7}$$

since ρ, p, u all $\to 0$ as $Y \to +\infty$. We write $u = \partial\phi/\partial Y$ and using (3.7) in (3.5) and (3.6) and eliminating ρ and p we obtain the equation

$$\left(1-\frac{\partial}{\partial Y}+\delta\frac{\partial}{\partial T}\right)\left(-2\delta\frac{\partial^2\phi}{\partial Y\,\partial T}+\delta^2\frac{\partial^2\phi}{\partial T^2}\right)+2\delta\frac{\partial^2\phi}{\partial Y^2}$$
$$=\epsilon\left(1-\frac{\partial}{\partial Y}\right)(\gamma_f+1)\frac{\partial\phi}{\partial Y}\frac{\partial^2\phi}{\partial Y^2}+O(\epsilon^2,\epsilon\delta). \qquad (3.8)$$

Rewritten in terms of x and t this is

$$\left(1+\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}\right)+2\delta\frac{\partial^2\phi}{\partial x^2}=\epsilon\left(1-\frac{\partial}{\partial x}\right)\left[(\gamma_f+1)\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}\right], \qquad (3.9)$$

where $u = \partial\phi/\partial x$ and this is the approximate equation whose solution we now consider. In particular, when $\delta^2 \ll \epsilon \lesssim \delta$, we may neglect terms of $O(\delta^2)$ in (3.8) and the equation becomes

$$\left(1-\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t}+\left(1+\frac{\gamma_f+1}{2}\epsilon u\right)\frac{\partial u}{\partial x}\right)=\delta\frac{\partial u}{\partial x}, \qquad (3.10)$$

which is the non-dimensional form of (1.9). For simplicity, this is the equation which we consider in §4, but a similar analysis can be applied to the more general equation (3.9).

It may be noted that the relaxation time τ need not be regarded as a strict constant. If we adopt the physically more realistic form

$$\tau = \tau(p,\rho)$$

and suppose that the partial derivatives of τ with respect to ρ and p are not unduly large, then for variations of the magnitude contemplated in p and ρ we should have

$$\tau \simeq \tau_0+(p-p_0)\left(\frac{\partial\tau}{\partial p}\right)_0+(\rho-\rho_0)\left(\frac{\partial\tau}{\partial\rho}\right)_0,$$

i.e. $$\tau \simeq \tau_0(1+O(\epsilon))$$

and the term of order ϵ may be neglected in the above approximation.

Steady solution of the model equation

Equation (3.10) possesses a self-preserving solution equivalent to that given in §2 for the exact equations. As before, we introduce the single dimensional variable $\xi = x-Ut$ (2.5), and apply the boundary conditions $u(+\infty) = 0$, $u(-\infty) = c$ and $du/d\xi(\pm\infty) = 0$. Then the dimensional form of (3.10) is

$$\left[1-U\tau\frac{d}{d\xi}\right]\left[\left(\frac{U^2}{a_1^2}-1\right)\frac{du}{d\xi}\right]+2\delta\frac{du}{d\xi}-(\gamma_f+1)\left[1-\tau U\frac{d}{d\xi}\right]\frac{u}{a_1}\frac{du}{d\xi}=0,$$

which integrates to give

$$U^2 = a_0^2+\frac{(\gamma_f+1)}{2}a_1 c$$

and $$\xi+\text{constant} = \frac{2\tau}{(\gamma_e+1)c}[A_1\ln(c-u)-B_1\ln u]. \qquad (3.11)$$

where A_1 and B_1 differ from the corresponding coefficients in (2.11) by $O(\epsilon\delta, \delta^2)$. Thus the steady dispersed shock solution is included as a solution of the model equation within the limits of relevant approximation.

As was the case with the exact equation, the profile given by (3.11) turns back on itself if $U > a_1$, and the solution for which $u \to 0$ as $\xi \to +\infty$ contains a discontinuous jump as shown in the bottom curve of figure 1. From the model equation alone it is not possible to locate the jump (as it was not from the full inviscid equations in § 2). Its position is to be found by considering the effects of viscosity and heat conduction, which predominate in a region where u changes so rapidly as to be discontinuous when they are ignored. This is done in § 5. The equivalent result, for the full equations, is that a Rankine–Hugoniot jump takes place at constant composition.

4. Solution of the model equation

We consider the specific problem of a piston pushed gradually into a semi-infinite tube containing gas in equilibrium. The piston is pushed in steadily so that its velocity tends to a uniform velocity $c = a_1\epsilon$ as $t \to \infty$ and we expect the steady state wave form (2.12) to be approached as $t \to \infty$. The equation (3.10) derived in the last section is valid provided both ϵ^2 and δ^2 are negligible and for simplicity we consider this case [although the more general equation (3.9) can be solved similarly].

The non-dimensional boundary conditions are taken as

(i) $$u = F'(t) \quad \text{on} \quad x = \epsilon F(t), \tag{4.1}$$

(ii) $$u = 0 \quad \text{on} \quad x = t, \tag{4.2}$$

where $F'(t) \to 1$ as $t \to \infty$, F increases monotonically and is such that no shocks form in the interior of the gas. The condition necessary for no shocks to form on the leading characteristic can be found by a method described by Jeffrey & Taniuti (1964) and is

$$F''(0) < \delta/\epsilon,$$

which we assume to hold.

The condition which allows a fully dispersed shock wave of form (2.12) to exist as $t \to \infty$ is

$$c = a_1\epsilon < 4\delta a_1/(\gamma_e + 1),$$

and so we consider piston velocities in the range

$$\delta^2 \ll \epsilon < 4\delta/(\gamma_e + 1), \tag{4.3}$$

where the lower limit is required for (3.10) to be valid.

The problem can be solved by the method of matched asymptotic expansions.† The solution depends on the size of t and we find the appropriate solutions for different times by stretching the independent variables.

(i) For $t = O(1)$, the flow is essentially frozen. The first approximation in ϵ, δ to (3.10) is

$$\left(-\frac{\partial}{\partial x} + 1\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = 0,$$

† The method used is similar to that applied to the solution of a weak viscous shock by Moran & Shen (1966).

with $u = 0$ on $x = t$ and $u = F'(t)$ on $x = 0$. This leads to

$$u = F'(t-x), \tag{4.4}$$

which is the first-order frozen solution given by taking $\delta = 0$ in the exact equations.

(ii) As t increases, we see from (4.4) that $u \to 1$ unless $(t-x)$ is $O(1)$. Thus we change to co-ordinates $y = t-x$ and $t = t_1/\delta$ to study this next region. Then (3.10) is

$$\left(1+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial t_1}-\frac{(\gamma_f+1)}{2}\frac{\epsilon}{\delta}u\frac{\partial u}{\partial y}\right) = -\frac{\partial u}{\partial y}. \tag{4.5}$$

If $\epsilon/\delta = O(1)$ the equation cannot be simplified and since no solution of this non-linear equation is known, no progress can be made. Some numerical solutions of

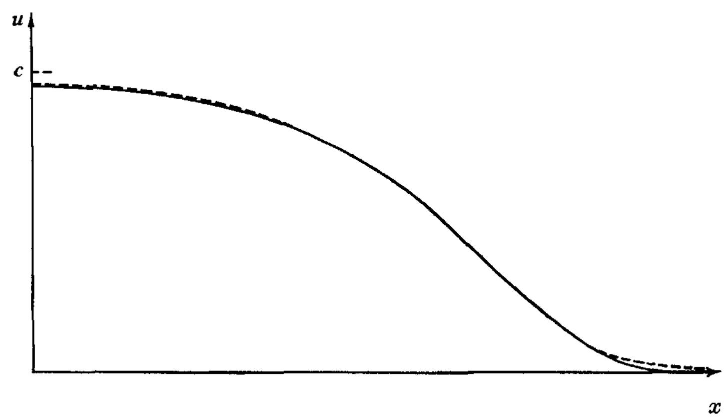

FIGURE 2. Numerical solution of equation (3.9) when $\epsilon \sim \delta$. $\delta = 0{\cdot}1$, $\epsilon = 0{\cdot}1$, $t = 100$. ——, numerical solution; – – – –, exact steady solution.

this equation have been calculated (see figure 2). However, if $\epsilon \ll \delta$, the equation can be simplified and solved. We have

$$\left(1+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial t_1}\right) = -\frac{\partial u}{\partial y}, \tag{4.6}$$

with $u = 0$ on $y = 0$ and $u = 1$ on $y = -\infty$. As $t_1 \to 0$, the matching condition with (4.4) is $u \to F'(y)$. Equation (4.6) is the telegraph equation and essentially the same as that solved by Moore & Gibson (1960) for weak disturbances in a relaxing gas.

The solution is

$$u = e^{-y-t_1}\int_0^y e^{\xi}(F'(\xi)+F''(\xi))\, I_0(2\surd[t_1(y-\xi)])\, d\xi, \tag{4.7}$$

where I_0 is the modified Bessel function of the first kind. This solution is a uniformly valid solution of (3.10) as $\epsilon \to 0$ for all $y > 0$.

By dividing the interval of integration and using different asymptotic representations for I_0 in each interval, we find the asymptotic value of u as $t_1 \to \infty$ which is

$$u \sim \tfrac{1}{2}\operatorname{erfc}\left(\frac{z}{2\surd t_1}\right) \quad \text{as} \quad t_1 \to \infty, \tag{4.8}$$

where $z = t_1 - y = x-(1-\delta)t$. Since $a_0 = a_1(1-\delta+O(\delta^2))$, we see that the wave which started close to the frozen Mach line is approaching the equilibrium Mach line $x = (a_0/a_1)t$ as t increases.

(iii) Solution (4.8) indicates that we should keep $z/\sqrt{t_1}$ of $O(1)$ as t increases in the next region and from equation (3.10) it can be seen that the appropriate scaling is

$$z_2 = \left(\frac{(\gamma_f+1)\epsilon}{2\delta}\right)z,$$

$$t_2 = \left(\frac{(\gamma_f+1)\epsilon}{2\delta}\right)^2 t_1,$$

and the first approximation to equation (3.10) is then

$$\left(\frac{\partial u}{\partial t_2}+u\frac{\partial u}{\partial z_2}\right) = \frac{\partial^2 u}{\partial z_2^2}, \tag{4.9}$$

with $u = 0, 1$ as $z \to \pm\infty$ and as $t_2 \to 0$, $u = \frac{1}{2}\operatorname{erfc}(z_2/2\sqrt{t_2})$ from (4.8). Equation (4.9) is Burgers' equation and can be solved with these boundary conditions. The solution is

$$u = 1\Big/\left[1+\frac{e^{\frac{1}{2}(z_2-\frac{1}{2}t_2)}\operatorname{erfc}(-z_2/2\sqrt{t_2})}{\operatorname{erfc}((z_2-t_2)/2\sqrt{t_2})}\right]. \tag{4.10}$$

As $t_2 \to \infty$ in (4.10),

$$u = \frac{1}{1+e^{\frac{1}{2}(z_2-\frac{1}{2}t_2)}}, \tag{4.11}$$

and so a steady-state wave form is obtained. The expression (4.11) is the first term in the expansion of the exact solution (2.12) and represents a wave travelling with velocity

$$\frac{dx}{dt} = \left(1-\delta+\frac{\gamma_f+1}{4}\epsilon\right).$$

The exact velocity of propagation from (2.10) with $c = a_1\epsilon$ is

$$U = \frac{\gamma_e+1}{4}\epsilon a_1 + a_0\left[1+\frac{\gamma_e+1}{4}\left(\frac{\epsilon a_1}{a_0}\right)^2\right]^{\frac{1}{2}}$$
$$= a_1\left(1-\delta+\frac{\gamma_f+1}{4}\epsilon+O(\epsilon\delta,\epsilon^2)\right).$$

Thus we can solve the problem completely when $\delta^2 \ll \epsilon \ll \delta$ and the solution is illustrated in figure 3. We find that all the above solutions of equation (3.10) are first-order solutions of the exact equations. Thus the approximations of § 3 are justified throughout the flow field and equation (3.10) is uniformly valid as $\epsilon \to 0$.

If the piston is slowly withdrawn (3.10) still holds and if $|\epsilon| \ll 1$ the solution can be found exactly as above. The problem is defined by (4.1) and (4.2) when $F'(\infty) = -1$. The solution is identical with that for $F'(\infty) = 1$ until the non-linear terms become important in the Burgers' equation region. The initial condition on (4.9) is now $u = -\frac{1}{2}\operatorname{erfc}(z_2/2\sqrt{t_2})$ as $t_2 \to 0$. The solution is then

$$u = -1\Big/\left[1+\frac{e^{-\frac{1}{2}(z_2+\frac{1}{2}t_2)}\operatorname{erfc}(-z_2/2\sqrt{t_2})}{\operatorname{erfc}((z_2+t_2)/2\sqrt{t_2})}\right].$$

The limit as $t_2 \to \infty$ of this expression has been considered by Jones (1964). Away from the lines $z_2 = 0,\ -t_2$ we have that as $t_2 \to \infty$,

$$u \sim \begin{cases} 0 & (0 < z_2), \\ z_2/t_2 & (-t_2 < z_2 < 0), \\ -1 & (z_2 < -t_2). \end{cases}$$

Since
$$\frac{\epsilon z_2}{t_2} = \frac{2}{\gamma+1}\left(\frac{x}{t} - (1-\delta)\right) = \frac{2}{\gamma+1}\left(\frac{x}{t} - \frac{a_0}{a_1}\right)$$

this solution is identical with (2.17) which is the equilibrium expansion fan as expected.

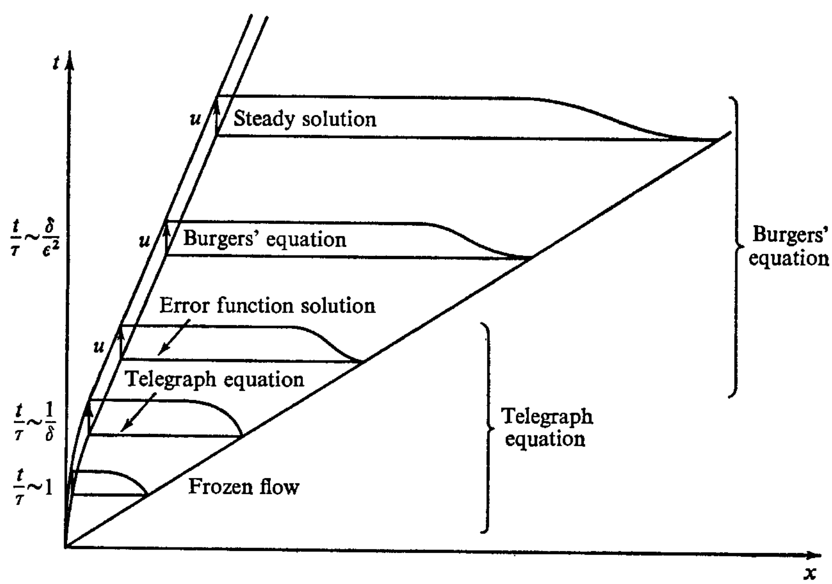

FIGURE 3. Velocity profiles when $\delta^2 \ll \epsilon \ll \delta \ll 1$.

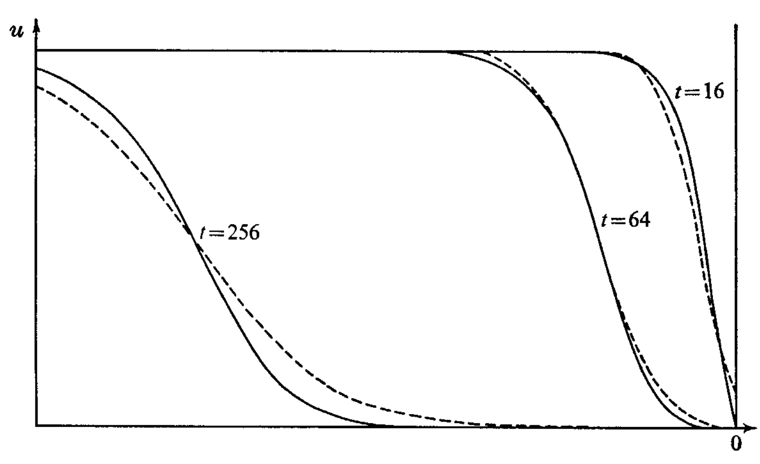

FIGURE 4. Comparison of numerical and analytic solution of (3.9) when $\epsilon \ll \delta$. $\delta = 0{\cdot}25$, $\epsilon = 0{\cdot}075$. ——, numerical solution; – – –, first-order solution.

It is now possible to consider solutions of the exact equations in the same regions and to compare results. Using the same stretched co-ordinates, we find that all the above solutions of (3.10) are first-order solutions of the exact equations. Thus the approximations of § 3 are justified throughout the flow field and (3.10) is uniformly valid as $\epsilon \to 0$. The solution can easily be extended to the case where $\epsilon \lesssim \delta^2$ and the form of the solution is just the same. Figure 4 compares the numerical solution of (3.10) with the above first-order solution when $\epsilon \ll \delta$.

5. Discontinuities

In previous sections we have considered the motion of the gas under the assumption that no discontinuities form in the flow. We now consider the shock relations which are appropriate in such a gas so that, in particular, the flow caused by an impulsively started piston may be examined.

A solution of the hyperbolic system of equations (2.1), (2.2) and (2.3) which involves discontinuities cannot be found uniquely without some further information. Lax (1954) has shown that there are an infinite number of 'permissible' weak solutions of such a system and to pick out the appropriate unique physical solution from this infinite set requires an extra physical condition. We suggest two ways to resolve this problem. The first is to use the Rankine–Hugoniot equations with the additional condition that the energy in the internal mode remains constant; or that the shock is 'frozen'. The second approach is to introduce viscous terms into the equations of motion and to find a continuous solution of these augmented equations.† This solution is uniquely determined and leads to a unique discontinuous solution of the inviscid equations in the limit as the viscosity tends to zero. We now treat the problem by this second method and obtain shock conditions which can be shown to be identical with those obtained on the frozen shock assumption. Since the problem we have in mind is that of a weak shock wave caused by an impulsive piston we first derive a uniformly valid approximate equation of the viscous equations by a similar method to that used in § 3. We then find a solution of this equation which is continuous across the shock.

The dimensional equations of motion including the 'first-order viscous terms' may be written:

$$\frac{d\rho}{dt}+\rho\frac{\partial u}{\partial x} = 0, \tag{5.1}$$

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{(\frac{4}{3}\mu_0+\mu_{v_0})}{\rho_0}\frac{\partial^2 u}{\partial x^2}, \tag{5.2}$$

$$\frac{de}{dt}+p\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{k_0}{\rho_0}\frac{\partial^2 T}{\partial x^2}, \tag{5.3}$$

where μ_0 is the equilibrium value of the viscosity, μ_{v_0} is that of the bulk viscosity and k_0 that of the thermal conductivity. These three equations, together with

† The steady solution of these equations has been described by Broer & Van den Bergen (1954).

(1.1) and (1.2), can now be treated exactly as in § 3 to derive an approximate equation similar to (3.10). We write

$$\mu = \frac{\frac{4}{3}\mu_0 + \mu_{v_0}}{a_1^2\rho_0\tau} \quad \text{and} \quad k = \frac{k_0}{R\rho_0 a_1^2\tau}$$

and assume that μ and k are small enough to neglect powers of μ, k and also terms in ϵk, $\epsilon\mu$, δk and $\delta\mu$. Since we require the limit as $\mu, k \to 0$ ultimately, this is not a restrictive assumption. The non-dimensional equation, derived on the additional assumption that ϵ^2, $\epsilon\delta$ and δ^2 are negligible, which is exactly analogous to (3.10) is

$$\left(1-\frac{\partial}{\partial x}\right)\left[\frac{\partial u}{\partial t}+\left(1+\frac{\gamma_f+1}{2}\epsilon u\right)\frac{\partial u}{\partial x}-\tfrac{1}{2}\lambda\frac{\partial^2 u}{\partial x^2}\right] = \delta\frac{\partial u}{\partial x}, \tag{5.4}$$

where

$$\lambda = \frac{\mu}{2}+\frac{(\gamma_f-1)^2 k}{\gamma_f}$$

is the non-dimensional 'diffusivity of sound'. The term involving λ will only be important inside the shock and so, regarding λ as the small parameter, we have a singular perturbation problem.†

We suppose that the shock is given by $dx/dt = V(t)$ and change to co-ordinates based on the shock:

$$\xi = x - \int_0^t V\,dt, \quad \eta = t.$$

Equation (5.4) is then

$$\left(1-\frac{\partial}{\partial \xi}\right)\left[\frac{\partial u}{\partial \eta}+\left(1-V+\frac{\gamma_f+1}{2}\epsilon u\right)\frac{\partial u}{\partial \xi}-\frac{\lambda}{2}\frac{\partial^2 u}{\partial \xi^2}\right] = \delta\frac{\partial u}{\partial \xi}. \tag{5.5}$$

To consider the shock region in detail, we stretch the variable ξ by putting $\xi = \lambda X$ so that, near the shock, the equation (5.5) is

$$\left(\lambda-\frac{\partial}{\partial X}\right)\left[\lambda\frac{\partial u}{\partial \eta}+\left(1-V+\frac{\gamma_f+1}{2}\epsilon u\right)\frac{\partial u}{\partial X}-\frac{1}{2}\frac{\partial^2 u}{\partial X^2}\right] = \lambda\delta\frac{\partial u}{\partial X}. \tag{5.6}$$

If we put $u = u_0+\lambda u_1+\ldots$ and $V = V_0+\lambda V_1+\ldots$, the first approximation in λ is

$$-\frac{\partial}{\partial X}\left[\left(1-V_0+\frac{\gamma_f+1}{2}\epsilon u_0\right)\frac{\partial u_0}{\partial X}-\frac{1}{2}\frac{\partial^2 u_0}{\partial X^2}\right] = 0,$$

so that

$$\left(1-V_0+\frac{\gamma_f+1}{2}\epsilon u_0\right)\frac{\partial u_0}{\partial X} = \frac{1}{2}\frac{\partial^2 u_0}{\partial X^2}, \tag{5.7}$$

since $\partial u_0/\partial X = \partial^2 u_0/\partial X^2 = 0$ at $X = \pm\infty$. For simplicity, we assume that the gas ahead of the shock is at rest, so that $u = 0$ at $X = \infty$. The value of u as $X \to -\infty$ is taken as u_s which is the value just behind the shock. Integrating the above equation again, we have

$$\frac{1}{2}\frac{\partial u_0}{\partial X}-\left(1-V_0+\frac{\gamma_f+1}{4}\epsilon u_0\right)u_0 = 0.$$

† Although we have neglected ϵ^2, δ^2 compared with λ, the shock relations we obtain are valid for any $\epsilon, \delta \ll 1$. The term in λu_{xx} can justifiably be retained when the length scale of the motion is $O(\lambda)$, which is the case inside a shock.

The conditions at $X = -\infty$ give

$$V_0 = 1 + \frac{\gamma_f + 1}{4} \epsilon u_s, \tag{5.8}$$

which agrees with the Rankine–Hugoniot value for the shock speed to the first order in ϵ. To determine the correct boundary conditions on an unsteady shock, we need to know both u and its normal derivative on the shock boundary. To find such a derivative we take the next approximation in (5.6) which gives on using (5.7)

$$-\delta \frac{\partial u_0}{\partial X} - \frac{\partial}{\partial X}\left(\frac{\partial u_0}{\partial \eta} + \left(1 - V_0 + \frac{\gamma_f + 1}{2}\epsilon u_0\right)\frac{\partial u_1}{\partial X} + \left(-V_1 + \frac{\gamma_f + 1}{2}\epsilon u_1\right)\frac{\partial u_0}{\partial X} - \frac{1}{2}\frac{\partial^2 u_1}{\partial X^2}\right) = 0.$$

Integrating this equation, we have

$$\frac{\partial u_0}{\partial \eta} + \left(1 - V_0 + \frac{\gamma_f + 1}{2}\epsilon u_0\right)\frac{\partial u_1}{\partial X} + \left(\frac{\gamma_f + 1}{2}\epsilon u_1 - V_1\right)\frac{\partial u_0}{\partial X} - \frac{1}{2}\frac{\partial^2 u_1}{\partial X^2} + \delta u_0 = 0. \tag{5.9}$$

We could now solve this equation for u_1 asymptotically as $X \to \pm\infty$ and then match the 'inner solution' in X with the 'outer' in ξ. However, since we only need the derivatives $\partial u_1/\partial X$, $\partial^2 u_1/\partial X^2$ to find the appropriate shock condition, the following method is shorter. We first take the 'difference' of (5.9) across the shock

$$\left[\frac{\partial u_0}{\partial \eta} + \delta u_0\right]_{X=-\infty}^{\infty} = \left[\frac{1}{2}\frac{\partial^2 u_1}{\partial X^2} - \left(1 - V_0 + \frac{\gamma_f + 1}{2}\epsilon u_0\right)\frac{\partial u_1}{\partial X}\right]_{X=-\infty}^{\infty}. \tag{5.10}$$

The matching procedure of Van Dyke (1964) is then used as follows:

(i) The two term inner solution for $\partial u/\partial X = \partial u_0/\partial X + \lambda\, \partial u_1/\partial X$, which may be expanded in terms of the outer variables

$$\frac{\partial u_0}{\partial X}(\pm\infty) + \lambda \frac{\partial u_1}{\partial X}(\pm\infty) + (\text{exponentially small terms}).$$

(ii) The two term outer solution for $\partial u/\partial X = \lambda\, \partial u/\partial \xi = \lambda\, \partial u_0/\partial \xi(\pm 0) + O(\lambda^2)$ when expanded in terms of the inner variable X. Matching these two expressions we have

$$\frac{\partial u_0}{\partial X}(\pm\infty) = 0,$$

$$\frac{\partial u_1}{\partial X}(\pm\infty) = \frac{\partial u_0}{\partial \xi}(\pm 0),$$

or
$$\frac{\partial u_1}{\partial X}(\pm\infty) = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial X}(-\infty) = \left(\frac{\partial u}{\partial \xi}\right)_s.$$

Similarly, consideration of $\partial^2 u/\partial X^2$, gives $\partial^2 u_1/\partial X^2(\pm\infty) = 0$.

In (5.10), these boundary conditions together with (5.8) give

$$\left(\frac{\partial u}{\partial \eta} + \frac{\gamma_f + 1}{4}\epsilon u \frac{\partial u}{\partial \xi} + \delta u\right)_s = 0.$$

Writing this in (x, t) variables we have

$$\frac{\partial u}{\partial t} + \left(1 + \frac{\gamma_f + 1}{4}\epsilon u\right)\frac{\partial u}{\partial x} + \delta u = 0 \tag{5.11}$$

as the appropriate boundary condition on the shock.

This shock condition enables us to solve the problem of an impulsively started piston. The solution is basically the same as that found in § 4 and is identical with that solution for all sufficiently large times.

6. Concluding remarks

The method described in §§ 3 and 4 may also be applied when the gas is subject to other dissipative effects. In particular, the effects of viscosity and heat conduction may be considered and then an analogous argument to that used in § 3 leads to Burgers' equation. The method has also been applied to a radiating gas when $\gamma - 1$ is small.

Mention should also be made of an analogy between the discontinuous shock structure due to relaxation when $U > a_1$ and the similar situation for shocks above a certain critical strength in a gas of zero Prandtl number, discussed *inter alia* by Hayes (1958, pp. 448–66). In this case too, we find that the fine structure of the discontinuity for a small but non-zero Prandtl number σ can be explored by a singular perturbation technique similar to that of the last section, the co-ordinates in the shock front being stretched with length scale σ.

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